# Finite Difference Solution to the Bagley-Torvik Equation

Aaron Larsen [1]

[1] Brigham Young University, Utah, USA

UCUR 2020/Utah State University/February 7, 2020

# Outline

1. The Fractional Derivative
2. Fractional Derivative Operators
3. The Conformable Fractional Derivative
4. Condition Number
5. Eigenvalues

# Why Fractional Derivatives?

- Used to model physical and engineering processes
- Used to model anomalous diffusion
- Economic growth modeling
- Model of the Ebola Hemorrhagic Fever

# The Fractional Derivative

General definition of the first derivative:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = D^1 f(x)$$

This definition can be applied to $D^n f(x)$ where $n \in \mathbb{N}$.

Since the 17th century, mathematicians have considered when $n$ is a non-integer. Thus began the study of fractional calculus.

# Fractional Derivative Operators

Caputo Operator

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx$$

Riemann-Liouville Operator

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

Where $\alpha \in [n-1, n)$

Note: $\Gamma(x) = (x-1)!$

# The Conformable Fractional Derivative

Khalil, et al. (2013) presented the following definition:

For $f : [0, \infty) \to \mathbb{R}$ and $t > 0$,

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{(1-\alpha)}) - f(t)}{\epsilon}$$

Where $\alpha \in (0, 1)$

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f^{(\lceil \alpha \rceil - 1)}(t + \epsilon t^{(\lceil \alpha \rceil - \alpha)}) - f^{(\lceil \alpha \rceil - 1)}(t)}{\epsilon}$$

Where $\alpha \in (n, n+1]$

This is a more local definition of the fractional derivative, as opposed to the global nature of the Riemann-Liouville and Caputo operators.

# Differences between the fractional derivative operators and the conformable fractional derivative

- Using the Riemann-Liouville operator, $D_a^\alpha(1) \neq 0$.
- The Caputo operator makes the assumption the the function $f$ is differentiable.
- All fractional derivatives do not satisfy the following derivative rules:
  - The derivative of the product of two functions.
  - The derivative of the quotient of two functions.
  - The chain rule.
- Unlike the Caputo and Riemann-Liouville operators, the conformable fractional derivative satisfies all of this.
- A sparse matrix can be used when using the conformable fractional derivative, thus being more computationally effective. The other operators require a dense matrix that is more complex the the prior.

# Relating the Conformable Derivative to Integer Derivatives

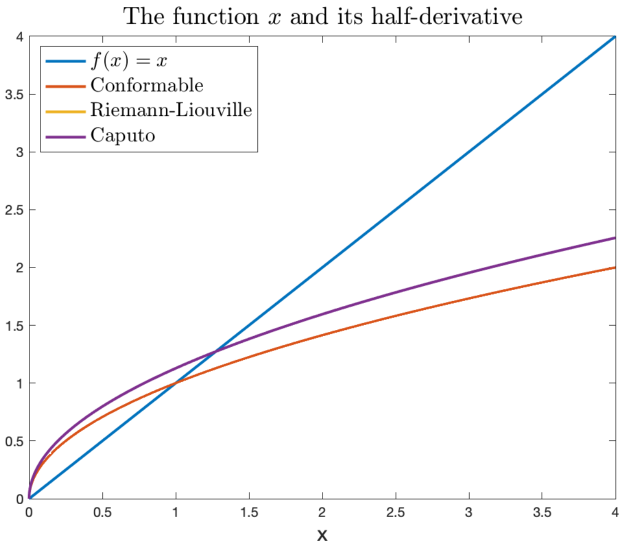
Let $\alpha \in (n, n+1]$. Let $h = \epsilon t^{(\lceil\alpha\rceil - \alpha)}$. Then $\epsilon = h t^{(1-\lceil\alpha\rceil+\alpha)}$.

$$
\begin{aligned}
T_\alpha(f)(t) &= \lim_{\epsilon\to 0} \frac{f^{(\lceil\alpha\rceil-1)}(t + \epsilon t^{(\lceil\alpha\rceil-\alpha)}) - f^{(\lceil\alpha\rceil-1)}(t)}{\epsilon} && (1)\\
&= \lim_{h\to 0} \frac{f^{(\lceil\alpha\rceil-1)}(t+h) - f^{(\lceil\alpha\rceil-1)}(t)}{h t^{(1-\lceil\alpha\rceil+\alpha)}} && (2)\\
&= t^{(\lceil\alpha\rceil-\alpha)} \lim_{h\to 0} \frac{f^{(\lceil\alpha\rceil-1)}(t+h) - f^{(\lceil\alpha\rceil-1)}(t)}{h} && (3)\\
&= t^{(\lceil\alpha\rceil-\alpha)} f^{\lceil\alpha\rceil}(t) && (4)
\end{aligned}
$$

Likewise, for $\alpha \in (0,1)$: $\quad T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$

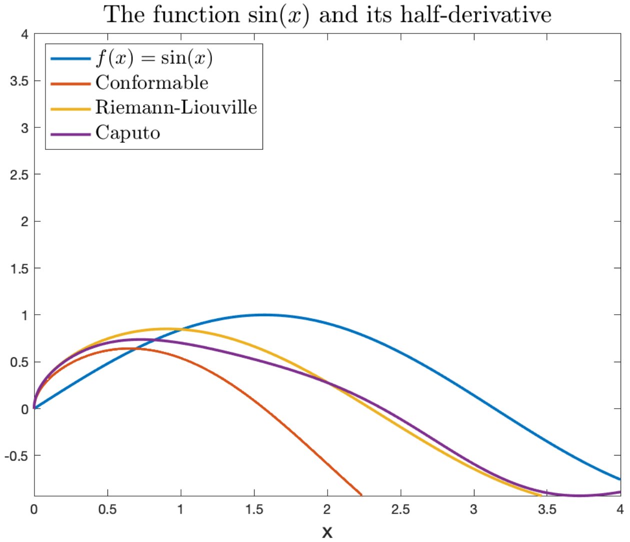
# Comparison with Other Fractional Derivative Operators

$f(x) = x \quad \alpha = \frac{1}{2}$



# Comparison with Other Fractional Derivative Operators

$f(x) = \sin(x) \quad \alpha = \frac{1}{2}$



# Application to the Bagley-Torvik Equation

The Bagley-Torvik Equation is of the form:

$$AD^2 f(t) + BD^{\alpha}(t) + Cf(t) = g(t)$$

where $\alpha = \frac{3}{2}$

This equation simulates the motion of a rigid plate immersed in a Newtonian fluid. This is also used to model viscoelastic fluids in a general setting.

# The Bagley-Torvik Equation Test Case

Using a version of the Bagley-Torvik Equation:

$$f^{''}(x) + f^{\frac{3}{2}}(x) + f(x) = g(x) \qquad f(0) = f(2) = 0$$

where $f(x) = x(2-x)e^{-x}$ is the exact solution.

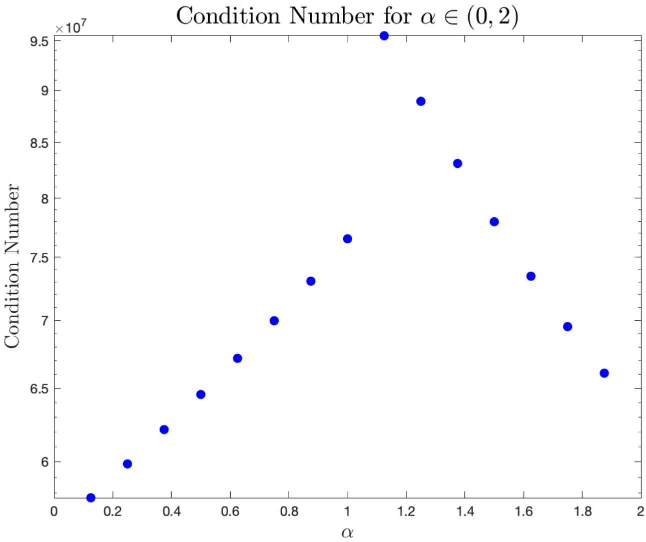
We will analyze the effect of the value of $\alpha$ on the eigenvalues of the matrix representation of the differential equation, as well as its effect on the condition number of the matrix.

# The effect on the eigenvalues when solving fractional differential equations

The eigenvalues of a matrix demonstrate its stability. The following graph is for a $10^4 \times 10^4$ matrix.

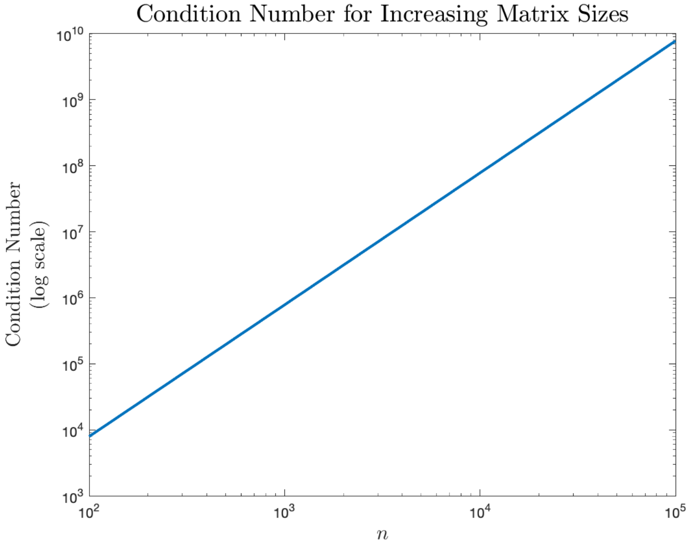
# The effect on the condition number with varying $\alpha$ values

The condition number measures how sensitive a function is to changes in the input. The following graph is for a $10^4 \times 10^4$ matrix.



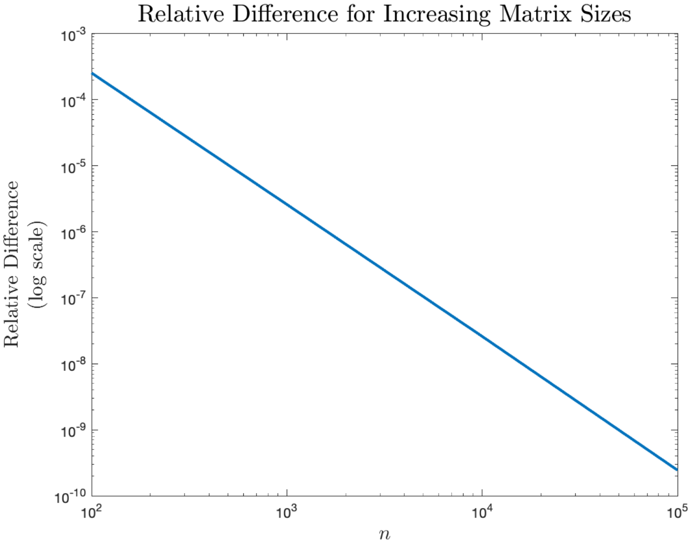
# The effect on the condition number with increasing matrix size

$$f''(x) + f^{\frac{3}{2}}(x) + f(x) = g(x) \qquad M_n(\mathbb{R})$$



# The effect on the relative difference with increasing matrix size

$f''(x) + f^{\frac{3}{2}}(x) + f(x) = g(x)$ Convergence Order $\approx 1.9957$



# Current Conclusions

1. The conformable derivative gives a rough estimate of the calculation
2. It provides a computationally cheap approximation of the fractional derivative